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JACOBIANS WITH A VANISHING THETA-NULL IN GENUS 4

BY

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ABSTRACT

In this paper we prove a conjecture of Hershel Farkas [11] that if a 4-dimensional principally polarized abelian variety has a vanishing theta-null, and the Hessian of the theta function at the corresponding 2-torsion point is degenerate, the abelian variety is a Jacobian.

We also discuss possible generalizations to higher genera, and an interpretation of this condition as an infinitesimal version of Andreotti and Mayer's local characterization of Jacobians by the dimension of the singular locus of the theta divisor.

Introduction

The study of the geometry of the theta divisor of Jacobians of algebraic curves is a very classical subject going back at least to Riemann's celebrated theta-singularity theorem itself, later strengthened by Kempf [21]. The geometry of the canonical curve was also shown to be related to the geometry of the theta

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divisor — in particular, Green's theorem [16] says that the tangent cones to the Jacobian theta divisor at its double points span the ideal of quadrics containing the canonical curve.

These tangent cone quadrics in fact have rank at most 4; it can also be shown that the singular locus, $\operatorname{Sing}\Theta$, of a theta divisor a Jacobian of a curve of genus g has dimension greater or equal to $g-4$ ($g-3$ for hyperelliptic curves). It was thus asked whether this is a characteristic property for $\mathcal{J}_g$ — the closure of the locus of Jacobians — within the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties.

This study was undertaken by Andreotti and Mayer [1]. Let $\mathcal{A}_g$ be the moduli space of (complex) principally polarized abelian varieties — **ppavs** for short. Denote by $N_k \subset \mathcal{A}_g$ the locus of ppavs for which $\dim \operatorname{Sing}\Theta \geq k$. Andreotti and Mayer showed that $\mathcal{J}_g$ is an irreducible component of N_{g-4}. The situation was further studied by Beauville [3], Debarre [6],[7], and Donagi [9]. It was shown that for $g \geq 4$ the locus N_{g-4} is reducible (and thus not equal to $\mathcal{J}_g$); however, conjecturally all components of N_{g-4} other than $\mathcal{J}_g$ are contained in the theta-null divisor θ_{null} (the zero locus of the product of all theta constants with half-integral characteristics; alternatively, the locus of those ppavs for which a symmetric theta divisor has a singularity at a 2-torsion point), which is a component of N_0, [24].

Thus it is natural to try to study the intersection of $\mathcal{J}_g$ with the other components of N_{g-4}, or at least with $\theta_{\text{null}} \subset N_0$. This is the object of this paper: we are interested in the locus within θ_{null} where the jet of the locus $\operatorname{Sing}\Theta$ at the corresponding 2-torsion point has dimension $g-4$, i.e., the rank of the tangent cone to the theta divisor at this point is 3. We prove H. Farkas' conjecture [11] that in genus 4 this condition characterizes Jacobians with a vanishing theta-null. Our approach may also be of use in approaching the same statement for arbitrary genus g.

In [20] Izadi used the Prym map to define for any 4-dimensional ppav a natural locus $T \subset |2\Theta|_{00}$ in the 2-theta linear system, and study its singularities (Theorem 2 in [20]), which appear to be the same as those of the theta divisor. We believe studying the relationship of the geometries of T and the theta divisor to be of interest. Recently Casalaina–Martin [5], by using the Prym construction, described the multiplicity of the singularities of theta divisor of ppavs of dimension 4 and 5. It would be interesting to try to combine his approach with ours to study the rank of the tangent cone to the theta divisor for $g=5$.

1. Notations and definitions

Definition 1: Let $\mathcal{H}_g$ denote the **Siegel upper half-space**, i.e. the set of symmetric complex $g \times g$ matrices τ with positive definite imaginary part. Each such τ defines a complex principally polarized abelian variety (ppav for short) $\mathbb{C}^g/\tau\mathbb{Z}^g + \mathbb{Z}^g$. If $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})$ is a symplectic matrix in the $g \times g$ block form, then its action on $\tau \in \mathcal{H}_g$ is defined by $\sigma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}$, and the moduli space of ppavs is the quotient $\mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$. We denote by $\mathcal{J}_g \subset \mathcal{A}_g$ the closure of the locus of Jacobians of Riemann surfaces of genus g.

A period matrix τ is called **decomposable** if there exists $\sigma \in \mathrm{Sp}(2g, \mathbb{Z})$ such that

$$\sigma \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \tau_i \in \mathcal{H}_{g_i},\ g_1 + g_2 = g,\ g_i > 0;$$

otherwise we say that τ is indecomposable.

Definition 2: For $\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g$, thought of as vectors of zeros and ones, $\tau \in \mathcal{H}_g$ and $z \in \mathbb{C}^g$, the **theta function with characteristic** $[\varepsilon, \delta]$ is

$$\theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i \left[{}^t(m + \varepsilon/2)\tau(m + \varepsilon/2) + 2\ {}^t(m + \varepsilon/2)(z + \delta/2) \right].$$

A **characteristic** $[\varepsilon, \delta]$ is called **even** or **odd** depending on whether $\theta\left[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}\right](\tau, z)$ is even or odd as a function of z, which corresponds to the scalar product $\varepsilon \cdot \delta \in \mathbb{Z}/2\mathbb{Z}$ being zero or one, respectively. A **theta constant** is the evaluation at $z = 0$ of a theta function. All odd theta constants of course vanish identically in τ.

Observe that

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\left(\tau, z + \tau\frac{\varepsilon}{2} + \frac{\delta}{2}\right) = \exp \pi i\Big(-\frac{{}^t\varepsilon}{2}\tau\frac{\varepsilon}{2} \ - \frac{{}^t\varepsilon}{2}\left(z + \frac{\delta}{2}\right)\Big)\theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z),$$

i.e., theta functions with characteristics are, up to some nonzero factor, the Riemann theta function (the one with characteristic $[0, 0]$) shifted by points of order two.

Consider a pair $\rho = (\rho_0, r)$, where $r \in \mathbb{R}$, and $\rho_0 : \mathrm{GL}(g, \mathbb{C}) \to \mathrm{End}\, V$ is an irreducible rational representation with the highest weight $(k_1, k_2, \ldots, k_g)$, $k_1 \geq k_2 \geq \cdots \geq k_g = 0$; we then use the notation

$$\rho(A) = \rho_0(A) \det A^{r/2}\ .$$

Definition 3: A map $f:\mathcal{H}_g\to V$ is called a **modular form for** ρ with respect to a finite index subgroup $\Gamma\subset \mathrm{Sp}(2g,\mathbb{Z})$ if

$$f(\sigma\cdot\tau)=\rho(c\tau+d)f(\tau)\quad \forall\tau\in\mathcal{H}_g,\ \forall\sigma\in\Gamma,$$

and if additionally f is holomorphic at all cusps of $\mathcal{H}_g/\Gamma$.

If ρ_0 is the trivial representation on $\mathbb{C}$, then we call this a **scalar modular form** of weight $r/2$.

For a finite index subgroup $\Gamma\subset \mathrm{Sp}(2g,\mathbb{Z})$ a **multiplier system** of weight $r/2$ is a map $v:\Gamma\to\mathbb{C}^*$, such that the map

$$\sigma\mapsto v(\sigma)\det(c\tau+d)^{r/2}$$

satisfies the cocycle condition for every $\sigma\in\Gamma$ and $\tau\in\mathcal{H}_g$ (note that the function $\det(c\tau+d)$ possesses a square root). Clearly a multiplier system of integral weight is a character.

Definition 4: A map $f:\mathcal{H}_g\to V$ is called a modular form for ρ, or simply a **vector-valued modular form**, if the choice of ρ is clear, **with multiplier** v, with respect to a finite index subgroup $\Gamma\subset \mathrm{Sp}(2g,\mathbb{Z})$ if

$$f(\sigma\cdot\tau)=v(\sigma)\rho(c\tau+d)f(\tau)\quad \forall\tau\in\mathcal{H}_g,\ \forall\sigma\in\Gamma, \tag{1}$$

and if additionally f is holomorphic at all cusps of $\mathcal{H}_g/\Gamma$.

Definition 5 (Theta constants are modular forms): We define the **level subgroups** of the symplectic group to be

$$\Gamma_g(n):=\left\{\sigma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in \mathrm{Sp}(2g,\mathbb{Z}):\ \sigma\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod n\right\}$$

$$\Gamma_g(n,2n):=\left\{\sigma\in\Gamma_g(n)\,|\,\mathrm{diag}(a^tb)\equiv\mathrm{diag}(c^td)\equiv 0 \bmod 2n\right\}.$$

These are finite index normal subgroups of $\mathrm{Sp}(2g,\mathbb{Z})$.

Under the action of $\sigma\in \mathrm{Sp}(2g,\mathbb{Z})$ the theta functions transform as follows:

$$\theta\left[\sigma\begin{pmatrix}\varepsilon\\ \delta\end{pmatrix}\right](\sigma\cdot\tau,\ {}^t(c\tau+d)^{-1}z)=\phi(\varepsilon,\,\delta,\,\sigma,\,\tau,\,z)\det(c\tau+d)^{\frac12}\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,\,z),$$

where

$$\sigma\begin{pmatrix}\varepsilon\\ \delta\end{pmatrix}:=\begin{pmatrix}d&-c\\-b&a\end{pmatrix}\begin{pmatrix}\varepsilon\\ \delta\end{pmatrix}+\begin{pmatrix}\mathrm{diag}(c\,{}^td)\\ \mathrm{diag}(a\,{}^tb)\end{pmatrix},$$

considered in $(\mathbb{Z}/2\mathbb{Z})^g$, and $\phi(\varepsilon,\ \delta,\ \sigma,\ \tau,\ z)$ is some complicated explicit function. For more details, we refer to [18] and [12].

Thus theta constants with characteristics are (scalar) modular forms of weight 1/2 with respect to $\Gamma_g(4,8)$, i.e. we have

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\sigma\cdot\tau,0)=\det(c\tau+d)^{1/2}\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,0)\quad\forall\sigma\in\Gamma_g(4,8).$$

Definition 6: We call the **theta-null divisor** $\theta_{\rm null}\subset\mathcal{A}_g$ the zero locus of the product of all even theta constants. We define a stratification of $\theta_{\rm null}$ as follows. For $h=0,\ldots,g$ we let

$$\theta^h_{\rm null}=\Big\{\tau\in\mathcal{H}_g:\exists[\varepsilon,\delta]\text{ even },\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)=0;\ \mathrm{rk}\left(\frac{\partial^2\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z)}{\partial z_j\partial z_k}\Big|_{z=0}\right)\le h\Big\},$$

i.e., the locus of points on $\theta_{\rm null}$ where the rank of the tangent cone to the theta divisor at the corresponding point $(\tau\varepsilon+\delta)/2$ of order two is at most h.

The partial derivatives of the theta function are not modular forms. However, the partial τ_{jk} derivative of a section $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)$ of a line bundle on $\mathcal{H}_g$, when restricted to the zero locus of this theta constant, is a section of the same bundle. By the heat equation this means that on the locus $\{\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)=0\}$ the second derivative $\frac{\partial^2\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z)}{\partial z_j\partial z_k}|_{z=0}$ is a modular form for $\Gamma_g(4,8)$.

By the above transformation formulas, we also see that $\theta_{\rm null}$ and $\theta^h_{\rm null}$ are well-defined on $\mathcal{A}_g$ and not only on the **level moduli spaces** $\mathcal{A}_g(4,8):=\mathcal{H}_g/\Gamma_g(4,8)$. Since the theta constant with characteristic is up to a non-zero factor the value of Riemann theta function at the corresponding 2-torsion point, $\theta_{\rm null}$ can also be described set-theoretically as the locus of ppavs for which Θ has a singularity at a 2-torsion point. The locus $\theta_{\rm null}$ is indeed a divisor (because theta constants do not vanish identically).

7 (The ring of modular forms): We recall that the theta constants define an embedding ([18], chapter V)

$$Th:\mathcal{A}_g(4,8)\to\mathbb{P}^{2^{g-1}(2^g+1)-1}$$
$$\tau\mapsto\left\{\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau,0)\right\}_{[\varepsilon,\delta]\ \text{even}},$$

which extends to the Satake compactification $\overline{\mathcal{A}_g(4,8)}$. Hence the ring of scalar modular forms for $\Gamma(4,8)$ is the integral closure of the ring $\mathbb{C}\left[\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]\right]$. The ideal of algebraic equations defining $Th(\overline{\mathcal{A}_g(4,8)}) \subset \mathbb{P}^{2^{g-1}(2^g+1)-1}$ is known completely only for $g \le 2$ (and almost known for $g = 3$, see [14]).

8: Since theta functions satisfy the heat equation

$$\frac{\partial^2 \theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z)}{\partial z_j \partial z_k} = \pi i (1+\delta_{j,k}) \frac{\partial \theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,z)}{\partial \tau_{jk}},$$

(where $\delta_{j,k}$ is Kronecker's symbol), the Hessian of a theta function with respect to z can be rewritten using the first derivatives with respect to τ_{jk}. Hence if a point $x = \tau\varepsilon/2 + \delta/2$ of order two is a singular point in the theta divisor, which is simply to say $\theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\tau,x) = 0 = \theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)$ (the first derivatives at zero of an even function are all zero), the rank of the quadric defining the tangent cone at x is the rank of the matrix obtained by applying the $g \times g$-matrix-valued differential operator

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial \tau_{11}} & \frac{1}{2}\frac{\partial}{\partial \tau_{12}} & \dots & \frac{1}{2}\frac{\partial}{\partial \tau_{1g}} \\ \frac{1}{2}\frac{\partial}{\partial \tau_{21}} & \frac{\partial}{\partial \tau_{22}} & \dots & \frac{1}{2}\frac{\partial}{\partial \tau_{2g}} \\ \dots & \dots & \dots & \dots \\ \frac{1}{2}\frac{\partial}{\partial \tau_{g1}} & \dots & \dots & \frac{\partial}{\partial \tau_{gg}} \end{pmatrix}$$

to $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)$.

2. Equations for θ_{null}^h

From now on, we will drop the argument $z = 0$ in theta functions, i.e., we will write $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)$ instead of $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau,0)$.

The locus θ_{null}^h is given by the conditions

$$\left\{\exists\ [\varepsilon,\delta] \text{ even};\ 0 = \theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau);\ \operatorname{rk}\mathcal{D}\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau) \le h\right\}.$$

We can get equations for θ_{null}^h by setting all $(h+1)\times(h+1)$ minors of $\mathcal{D}\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)$ equal to zero, but these minors are not modular forms: the derivative of a section of a bundle is only a section of that bundle when restricted to the zero set of the section, i.e. $\mathcal{D}\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)$ is not a modular form, but is modular when restricted to the locus $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau) = 0$. These $(h+1)\times(h+1)$ minors of $\mathcal{D}$ are not invariant under $\mathrm{Sp}(2g,\mathbb{Z})/\Gamma_g(4,8)$, and thus for technical reasons we will work on $\mathcal{A}_g(4,8)$ instead of $\mathcal{A}_g$. However, the locus θ_{null}^h that we are describing is invariant under the action of $\mathrm{Sp}(2g,\mathbb{Z})$, and we will thus be able to easily descend from $\mathcal{A}_g(4,8)$ to $\mathcal{A}_g$ by symmetrizing.

The divisor $\theta_{\rm null} \subset \mathcal{A}_g(4,8)$ is reducible. Its irreducible components are the divisors of individual theta constants with characteristics (cf. [13] p. 88 for $g \geq 3$; it is easily verified also for $g = 1, 2$). These components are all conjugate under the action of $\mathrm{Sp}(2g, \mathbb{Z})$, and thus for our purposes we can restrict to one component. Without loss of generality we can take this to be $\theta_0 := \{\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau) = 0\} \subset \mathcal{A}_g(4,8)$, and consider its stratification, letting θ_0^h be the set of those $\tau \in \theta_0$ for which the rank of the tangent cone, i.e. of the Hessian of the theta function, at zero is at most h (we take this set with the reduced scheme structure).

Following the ideas of [17], we observe that for any even characteristic $[\varepsilon, \delta] \neq [0, 0]$ the expression

$$\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)^2\mathcal{D}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}/\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}\right)(\tau) = (1+\delta_{j,k})\left[\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\frac{\partial\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau)}{\partial\tau_{jk}} - \theta\begin{bmatrix}0\\0\end{bmatrix}(\tau)\frac{\partial\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)}{\partial\tau_{jk}}\right]$$

is a vector-valued modular form of weight 1 for $\rho_0 = (2, 0, \ldots, 0)$ with respect to $\Gamma_g(4,8)$.

We denote by $\mathcal{B}_h\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]\right)(\tau)$ the $\binom{g}{h} \times \binom{g}{h}$ symmetric matrix obtained by taking in lexicographic order all the $h \times h$ minors of the above matrix. $\mathcal{B}_h$ is then a vector-valued modular form of weight h with $\rho_0 = (2, \ldots, 2, 0 \ldots, 0)$, with respect to $\Gamma_g(4,8)$.

THEOREM 9: *As a set, the locus θ_0^h is defined by*

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tau, 0) = 0 = \mathcal{B}_{h+1}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}\right)(\tau) \quad \forall[\varepsilon, \delta] \neq [0, 0] \text{ even.}$$

Proof. One implication is trivial. Conversely, let us assume that all equations are satisfied; there always exists an even characteristic $[\varepsilon, \delta]$ such that $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau) \neq 0$, thus $\mathcal{B}_{h+1}\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right]\right)(\tau) = 0$ implies $\tau \in \theta_0^h$. ∎

3. The main theorem

Our main result of this section is the proof of the following conjecture of H. Farkas [11]:

THEOREM 10: *If for $\tau \in \mathcal{A}_4$, some theta constant and its Hessian are both equal to zero, this ppav is a Jacobian, i.e.*

$$\theta_{\rm null}^3 = \mathcal{J}_4 \cap \theta_{\rm null}.$$

Proof. The fact that this vanishing holds for Jacobians (i.e. the inclusion $\supseteq$) is well-known. Indeed, given a Jacobian with a vanishing theta-null, i.e., such that some 2-torsion point x lies on the theta divisor, it is a consequence of Kempf's singularity theorem (see [21], [2]) that the rank of the Hessian of the theta function (i.e. of the tangent cone) at x is equal to three.

Remark 11: Beauville proves ([3], Proposition 7.5), by exhibiting such a ppav as an appropriate Prym, that for a generic ppav in $\theta_{\rm null} \subset \mathcal{A}_4$ the rank of the Hessian of the theta function at the corresponding 2-torsion point is equal to 4, i.e., that a general point in $\theta_{\rm null}$ does not lie in $\theta^3_{\rm null}$. It, of course, follows then that a generic ppav in $\theta_{\rm null}$ is not in $\mathcal{J}_4$. However, to prove the theorem, one needs to prove that any ppav in $\theta_{\rm null} \setminus \mathcal{J}_4$ is not in $\theta^3_{\rm null}$, which would require also working with the Prym map for possibly singular curves, and thus appears quite hard.

Since $\mathrm{Sp}(8,\mathbb{Z})/\Gamma_4(4,8)$ acts transitively on the set of theta constants with characteristics, it is enough to take $[\varepsilon,\delta]=[0,0]$ above, and show (here θ denotes the theta function with zero characteristics) that if $\theta(\tau)=\det \mathcal{D}\theta(\tau)=0$, then $\tau \in \mathcal{J}_4 \cap \theta_{\rm null}$.

We will denote $\mathcal{J}_4(4,8) \subset \mathcal{A}_4(4,8)$ the Jacobian locus, and also denote $J := \overline{Th(\mathcal{J}_4(4,8))} \subset A := \overline{Th(\mathcal{A}_4(4,8))}$, and denote the chosen component of the theta-null by $T := A \cap \{\theta(\tau)=0\}$. Let us denote the locus we are interested in by $S := T \cap \{\det \mathcal{D}\theta(\tau)=0\}$.

Theorem 10 is then the statement that $S = J \cap T$, of which we already know the inclusion $S \supseteq J \cap T$ from the previous discussion.

Lemma 12: $\dim S = \dim(J \cap T) = 8$.

Proof. It is known that no theta constant vanishes identically on the Jacobian locus: the divisor of any one theta constant is irreducible in $\mathcal{A}_4(4,8)$ ([13], p. 88), and thus cannot contain the Jacobian locus, which is itself a divisor. Thus $J \cap T$ is strictly contained in J and is given by one equation, so $\dim(J \cap T) = \dim J - 1 = 8$. On the other hand, Beauville showed that the locus S is strictly contained in T (by showing that a general point of T does not lie in S — see Remark 11), and since locally in $\mathcal{H}_4$ it is given by one extra equation, we get $\dim S = \dim T - 1 = \dim A - 2 = 8$. ∎

By the discussion in the previous section we know that $S \supseteq J \cap T$, and to prove that $S = J \cap T$ it is enough to show that the degrees of the two sets, as 8-dimensional subvarieties of the projective space, are equal. We will now compute these degrees; in doing so we will need to be careful to distinguish the degree of the scheme $\deg X$ from the degree of the underlying set $\deg X_{\mathrm{red}}$.

LEMMA 13: *The degree* $\deg(J \cap T)_{\mathrm{red}}$ *is equal to* $8 \deg A$*, and degree of the scheme* $J \cap T$ *is equal to* $16 \deg A$*.*

Proof. Since $\{\theta(\tau) = 0\}$ is a hyperplane in $\mathbb{P}^{135}$, we have $\deg T = \deg A$, and $\deg(J \cap T) = \deg J$. Recalling that A and J are both irreducible varieties, and $J \subset A$ is given by one equation of degree 16 in theta constants [14], i.e. by a polynomial of degree 16 in the coordinates of $\mathbb{P}^{135}$, we see that $\deg J = 16 \deg A$.

However, an equation defining $J \subset A$ in the notation of [14] is

$$(r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)(r_1 - r_2 - r_3) = 0,$$

where each r_j is a monomial of degree 4 in theta constants.[1] Given $\tau \in T$, let us choose, without loss of generality, the equation above such that the theta constant that vanishes at T is one of the factors of, say, r_3, so that we have $r_3(\tau) = 0$. In this case the above equation becomes $(r_1 - r_2)^2(r_1 + r_2)^2 = 0$, so that the intersection $J \cap T$ has multiplicity at least two at τ. To finish the proof of the lemma, we need to show that the multiplicity is exactly two. This follows from Theorem 4.1 in [7], but for completeness let us give a direct proof using an argument similar to the one used in [19] to prove local irreducibility of the Schottky divisor.

Indeed, take a diagonal period matrix $\tau_0 \in \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_1$ with diagonal entries $\omega_1, \omega_2, \omega_3, \omega_4$, and denote by $\mathcal{O}$ the analytic local ring of $\mathcal{H}_4$ at τ_0. We shall use $\tau_{jj} - \omega_j$ for $1 \le j \le 4$, and $2\pi\sqrt{-1}\tau_{jk}$ for $1 \le j < k \le 4$ as generators of the maximal ideal $\mathfrak{m} \subset \mathcal{O}$. If we arrange the $2\pi\sqrt{-1}\tau_{ij}$ in the order $(12), (34), (13), (24), (14), (23)$ and call them $x_1, x_2, \ldots, x_6$, then the expression $(r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)(r_1 - r_2 - r_3)$ in $\mathfrak{m}^8/\mathfrak{m}^9$ is

[1] There are many possible choices for r_j, and thus many resulting forms of the equation, which are all conjugate under the action of $\mathrm{Sp}(8, \mathbb{Z})/\Gamma_4(4, 8)$. What happens is that $A \subset \mathbb{P}^{135}$ is itself given by a large number of equations (explicitly unknown to this date), and when we intersect A with any equation of the above form, the intersection is always the same, and in particular invariant under $\mathrm{Sp}(8, \mathbb{Z})/\Gamma_4(4, 8)$.

equal to

$$2^{16}\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)\delta(\omega_4)P(X),$$

where $\delta(\omega)$ is the unique cusp form of weight 12 for $\mathrm{SL}(2,\mathbb{Z})$, suitably normalized, and (see [19], p. 543)

$$P(X) = (x_1x_2 - x_3x_4)^2(x_5x_6)^2 - 2(x_1x_2 + x_3x_4)\prod_{j=1}^{6} x_j + \left(\prod_{j=1}^{4} x_j\right)^2.$$

For simplicity we choose, for this lemma only, the vanishing theta constant to be $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)$ with $\varepsilon = \delta = (1\,1\,0\,0)$, instead of the zero characteristic. This theta constant in $\mathfrak{m}/\mathfrak{m}^2$ has local expression x_1. Hence, by substitution, we have the expression $(x_3x_4x_5x_6)^2$ for $P(X)$, so the multiplicity of the intersection of T and J (as given by the Schottky relation) is exactly two. As a consequence, $\deg(J\cap T) = 2\deg(J\cap T)_{\mathrm{red}}$. ∎

To compute $\deg S$, we use results from the previous sections. The problem with dealing with S is that $\det\mathcal{D}(\theta(\tau))$ is not a modular form, and thus its zero locus is not well-defined on $\mathcal{A}_g(4,8)$. However, we can apply Theorem 9 for $g = 4, h = 3$.

Lemma 14: *Define a function F on $\mathcal{H}_g$ by*

$$F(\tau) := \left(\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right)^{2g} \det\mathcal{D}\left(\frac{\theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\tau)}{\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau)}\right).$$

Then for any even $[\varepsilon,\delta]$, scheme-theoretically we have

$$T\cap\{F(\tau)=0\} = T\cap\left\{0 = \left(\theta\begin{bmatrix}\varepsilon\\ \delta\end{bmatrix}(\tau)\right)^{g}\det\mathcal{D}\theta(\tau)\right\} \tag{2}$$

Proof. This is obtained by writing out the derivatives involved in F and using $\theta\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right](\tau) = 0$ on T. ∎

As discussed in [17], $F(\tau)$ is a scalar modular form of weight $g+2$ with respect to $\Gamma_g(4,8)$. We can now compute $\deg S$ and thus finish the proof of the theorem.

Lemma 15: *The degree* $\deg S_{\mathrm{red}}$ *is equal to* $8\deg A$ $(=\deg(J\cap T)_{\mathrm{red}})$.

Proof. For $g = 4$ the modular form F is of weight 6, while theta constants are of weight 1/2. Thus the divisor of F is numerically equivalent to the divisor of the 12th power of a theta constant. Hence the degree of the zero locus of F in $\mathbb{P}^{135}$ is equal to 12. Thus we have $\deg(T \cap \{F(\tau) = 0\}) = 12 \deg T = 12 \deg A$.

To understand the locus $T \cap \{F(\tau) = 0\}$, note that on the right-hand-side of equation (2) we have the union of two loci: S, which is exactly $T \cap \{\det \mathcal{D}\theta(\tau) = 0\}$, and $T \cap \{\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau) = 0\}$, with multiplicity $g = 4$. Of course the latter is the intersection of A with two hyperplanes, so its degree is equal to $\deg A$. Thus comparing the degrees on both sides of the equality in the previous lemma, we get

$$\deg S = 12 \deg A - 4 \deg A = 8 \deg A = \deg(J \cap T).$$

Since $\dim S = \dim(J \cap T)$, and we know that $S \supseteq J \cap T$, it follows that the scheme S is reduced, and thus $\deg S = \deg S_{\mathrm{red}}$. ∎

Since we have shown that $\deg S = \deg(J \cap T)_{\mathrm{red}}$, and we know $S \supseteq (J \cap T)$, Theorem 10 — the statement that the set $S = J \cap T$ — finally follows. ∎

As an immediate consequence we have the following

COROLLARY 16: *If for $\tau \in \mathcal{A}_4$ some theta constant vanishes: $\theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau) = 0$, and all 3×3 minors of the Hessian $\left.\frac{\partial^2 \theta\left[\begin{smallmatrix}\varepsilon\\ \delta\end{smallmatrix}\right](\tau, z)}{\partial z_j \partial z_k}\right|_{z=0}$ are zero, then τ has reducible theta divisor.*

Proof. By Theorem 10 we know that $\tau \in \mathcal{J}_4$. In this case the tangent cone at $0 \in \mathbb{C}^g / \tau\mathbb{Z}^g + \mathbb{Z}^g$ has rank two, hence τ cannot be a Jacobian. Thus τ is the period matrix of a decomposable ppav, cf [19]. ∎

Note that in trying to approach the problem in higher genus one trouble with this corollary is that the locus of decomposable ppavs within $\mathcal{A}_g$ is not contained in the closure of the Jacobian locus, while for genus 4 this is the case, as we have $\mathcal{J}_g = \mathcal{A}_g$ for $g = 1, 2, 3$.

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